Solution of transient viscoelastic flow problems approximated by a VMS stabilized finite element formulation using time-dependent subrid-scales

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Summary Recent studies indicate that classical residual-base stabilized methods for unsteady incompressible flows may experience difficulties when the time step is small in relation with the spatial grid size. The aim of this work is the design of finite element stabilized techniques based on the Variational Multiscale (VMS) method that allow to compute time-dependent viscoelastic flow problems with high elasticity and considering an anisotropic space-time discretization. Although the main advantage is achieve stable solutions for anisotropic space-time discretizations, other benefits related with elastic problems are proved in this study. In particular, the proposed methods are designed for the standard and logarithmic formulations in order to deal with high Weissenberg number problems, ensuring stability in all cases. A comparison between formulations and stabilization techniques will be performed to demonstrate the efficiency of time-dependent sub-grid scales and the term-by-term methodologies.

INTRODUCTION

Bochev et al. [1] argue that spatial stabilization in conjunction with finite differencing in time implies destabilizing terms and that $\delta t > Ch^2$ (where δt is the time step size, C a positive constant and h the spatial grid size) is a sufficient condition to avoid instabilities. Nevertheless, for anisotropic space-time discretizations, this inequality is not necessarily satisfied, and in fact complications in residual-based stabilized methods are reported. These problems can happen, for instance, when small time steps result from the necessity of accuracy to solve transient problems due to the presence of non-linear terms in the differential equations, a very common issue in viscoelastic flow formulations. In particular, the approximations used in Variational Multiscale (VMS) methods [2] usually neglect the time derivative of the sub-grid scales, resulting in the inequality $\delta t > Ch^2$ being required to obtain stable solutions. Consequently, anisotropic space-time discretizations cannot guarantee stability, as it is argued by Codina et al. [3]. By following these ideas, the present work pursues to expand transient subgrid-scale methods to the viscoelastic flow problem, such as it is presented in [4] for the Navier-Stokes incompressible problem using a split term-by-term method. The computation of viscoelastic flows leads to its own difficulties, especially when elasticity becomes dominant, i.e., when the dimensionless number known as the *Weissenberg number* is high: the so called High Weissenberg Number Problem (HWNP). A new formulation was proposed by Fattal and Kupferman in order to deal with these shortcomings: the so called Logarithmic Conformation Representation. In this sense, in [5] the authors apply this reformulation using a stabilized formulation based on the VMS method.

VISCOELASTIC FLOW PROBLEM

The governing equations for the viscoelastic flow problem in incompressible and isothermic conditions, are the conservation of momentum and mass and a constitutive equation which can be expressed as

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nabla \cdot \boldsymbol{T} + \nabla p = \boldsymbol{f} \text{ in } \Omega, t \in]0, t_{\mathrm{f}}[, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega, t \in]0, t_{\mathrm{f}}[, \tag{2}$$

$$\frac{1}{2\eta_p}\boldsymbol{\sigma} - \nabla^s \boldsymbol{u} + \frac{\lambda}{2\eta_p}\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \nabla \boldsymbol{u} - (\nabla \boldsymbol{u}^T) \cdot \boldsymbol{\sigma}\right) = \boldsymbol{0}, \text{ in } \Omega, t \in]0, t_{\mathrm{f}}[, \tag{3}$$

where Ω is the domain considered of $\mathbb{R}^d$ (d=2 or 3), whose boundary is $\partial\Omega$, during the time interval $[0, t_{\mathrm{f}}]$, where ρ denotes the constant density, p is the pressure field, $\boldsymbol{u}$ is the velocity field, $\boldsymbol{f}$ is the force field and $\boldsymbol{T}$ is the deviatoric stress tensor. In general, $\boldsymbol{T}$ is defined in terms of a viscous and a viscoelastic contribution as $\boldsymbol{T} = 2\eta_e \nabla^s \boldsymbol{u} + \boldsymbol{\sigma}$. Effective (or solvent) viscosity is denoted by η_e and the polymeric viscosity by η_p. The third equation of the system is the constitutive equation for the viscoelastic stress tensor. We have considered the Oldroyd-B model where λ is the relaxation time. The logarithmic reformulation of the equations is derived basically from a change of variables, where the stress tensor is replaced by $\boldsymbol{\sigma} = \dfrac{\eta_p}{\lambda_0}(\exp(\boldsymbol{\psi}) - \mathbf{I})$ in (1), (2) and (3). Particularly, λ_0 is linearly dependent with λ and is defined as $\lambda_0 = \max\{k\lambda, \lambda_{0,\min}\}$, being $k \leq 1$ a constant and $\lambda_{0,\min}$ a given threshold. Note that the complete development employed is extensively explained in [5].

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P1 elements	**Time step (δt)**			
Method	0.050	0.0250	3.125×10^{-3}	1.562×10^{-3}
Static-OSS	Solved	Failed	-	-
Dyn-OSS	Solved	Solved	Solved	Solved
Static-SOSS	Solved	Solved	Solved	Failed
Dyn-SOSS	Solved	Solved	Solved	Solved

Table 1: Solved and failed cases We $= 0.125$, $\alpha_{1,\min} \approx 1.156 \times 10^{-3}$.

	Weissenberg (We)			
Formulation	0.125	0.165	0.25	0.5
Std-Static	Solved	Failed	-	-
Std-Dyn	Solved	Solved	Solved	Failed
Log-Static	Solved	Solved	Failed	-
Log-Dyn	Solved	Solved	Solved	Solved

Table 2: Solved and failed cases for S-OSS formulations,dynamic and quasi-static, $\delta t = 0.1$.

STABILIZED FINITE ELEMENT FORMULATION

The stabilization method departs from the framework described in [2], which consists in splitting the unknowns $\boldsymbol{U}$ in the sum of two components, $\boldsymbol{U}_h$ (component which can be captured by the finite element space) and $\tilde{\boldsymbol{U}}$ (the remainder, called sub-grid scale). Let us suppose that $\mathcal{L}(\hat{\boldsymbol{u}}; \cdot)$ is a linear operator for a given $\hat{\boldsymbol{u}}$. Introducing the sub-grid scale decomposition and integrating by parts, the method leads to find $\boldsymbol{U}_h :]0, t_\mathrm{f}[\longrightarrow \boldsymbol{\mathcal{X}}_h$ such that

$$\mathcal{G}(\boldsymbol{U}_h, \boldsymbol{V}_h) + B(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h) + \sum_K \langle \tilde{\boldsymbol{U}}, \mathcal{L}^*(\boldsymbol{u}_h; \boldsymbol{V}_h) \rangle_K = L(\boldsymbol{V}_h), \tag{4}$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h$, where B is the bilinear form of the problem and $\mathcal{G}$ the temporal terms, $\mathcal{L}^*(\boldsymbol{u}_h; \boldsymbol{V}_h)$ is the formal adjoint of the operator $\mathcal{L}(\hat{\boldsymbol{u}}; \cdot)$ typically without considering boundary conditions, $\tilde{\boldsymbol{U}}$ is the sub-grid scale, which needs to be approximated and has components $\tilde{\boldsymbol{U}} = [\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}]$. Particularly, $\mathcal{L}(\hat{\boldsymbol{u}}; \cdot)$ is the operator associated to the viscoelastic flow problem and $L(\boldsymbol{V}_h)$ comes from the RHS terms in (1) - (3). Once operators $\mathcal{G}$ and $\mathcal{L}$ are defined for both formulations, the sub-grid scales are now the solution of this equation, written in terms of the finite element component:

$$\mathcal{D}_t(\tilde{\boldsymbol{U}}) + \boldsymbol{\alpha}^{-1}\tilde{\boldsymbol{U}} = \tilde{P}[\boldsymbol{F} - \mathcal{G}(\boldsymbol{U}_h) - \mathcal{L}(\boldsymbol{u}_h; \boldsymbol{U}_h)], \tag{5}$$

where we denote $\tilde{P}$ as the L^2 projection onto the space of sub-grid scales, $\mathcal{D}_t(\tilde{\boldsymbol{U}})$ is defined as the temporal derivative of the sub-grid scale and $\boldsymbol{\alpha}$ is taken as a diagonal matrix of stabilization parameters. Apart from the purely residual-based stabilization (denoted by OSS), we propose a term-by term stabilization motivated by the fact that not all the terms of some products provide stability, denoted by S-OSS. The idea of this method is to replace the solution of (5) in (4) but keep only the terms of the form one operator term applied to the unknown by the same operator term applied to the test function, thus neglecting the products of different operators.

NUMERICAL RESULTS AND CONCLUSIONS

The flow over a cylinder problem is used to achieve several objectives: firstly, to compare the various stabilization methods proposed (dynamic and quasi-static formulations) in terms of stability when the time step is small, and when the Weissenberg number increases. In first case, Table 1 shows us that the dynamic method is the most efficient. In the second study, Table 2 we can conclude that dynamic formulations are more effective avoiding elastic instabilities, allowing the computation of fluid flows with a higher Weissenberg number.

The results obtained are particularly remarkable due to the high Weissenberg number reached with the dynamic formulation, which remains stable even if the standard formulation is considered, apart from evident benefits in anisotropic space-time discretizations when the time step is small. The combination of dynamic sub-grid scales and the logarithmic formulation is capable of solving problems with higher elasticity.

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