UNIVERSITÀ DEGLI STUDI DI PADOVA

TUM IAS

CMN 2022, Las Palmas de Gran Canaria

A VMS-stabilized mixed formulation for non-linear incompressible solid mechanics problems using the implicit Material Point Method

Laura Moreno, Alessandro Contri and Antonia Larese

September, 2022

Outline

Motivation

The climatic change is one of the main causes of **water-related extreme events**.

Figure: **Landslide** in Wenchuan area of China (left) and bridge failed during Washington **floods** (right)

In recent years, **Material Point Method** (MPM) receive attention in geotechnical engineering.

Goal of the work

The development of a stabilized formulation for incompressible materials in the **MPM framework**. In particular:

1. **Incompressible solid mechanics in mixed formulation** for simulating large deformation regimes and
2. **Newtonian law** for a linearized displacement-based formulation.

All implementations are performed using **KRATOS Multiphysics**, an open source and high performance simulation software.

Why using the Material Point Method (MPM)?

FEM

- ✗ Large deformations regimes.

Discrete approaches

- ✗ Large scale problems.
- ✗ Complex material laws.

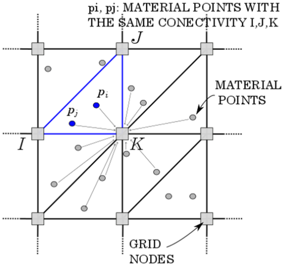
The Material Point Method

- ✓ Large deformation problems
- ✓ Free surface evolution
- ✓ Mass conservation.
- ✓ Complex material laws.

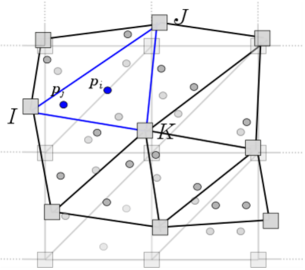
MPM is a particle-in-cell (PIC) method which takes advantage of all the potential and the well-established knowledge reached in **FE technology**.

The Material Point Method

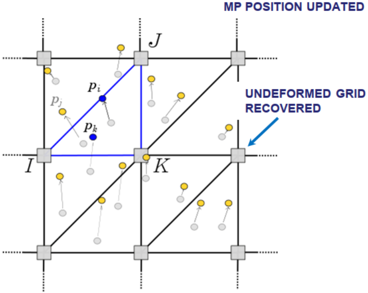
1. A background grid (fixed) used for the FEM solution system.
2. A collection of Material Points (MP) (Lagrangian).



Stage 1. Initialization



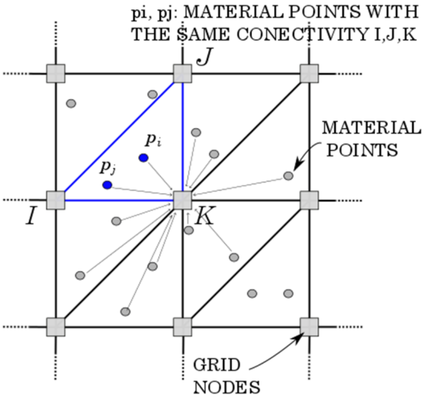
Stage 2. Calculation



Stage 3. Convection

Stage 1. Initialization phase

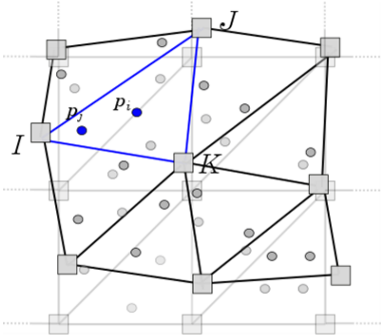
Definition of the initial conditions on the FE grid's nodes.



1. Extrapolation on the nodes.
2. Prediction of nodal displacement, velocity and acceleration.

Stage 2. Calculation phase

Update Lagrangian-FEM solution on the nodes of the background grid.



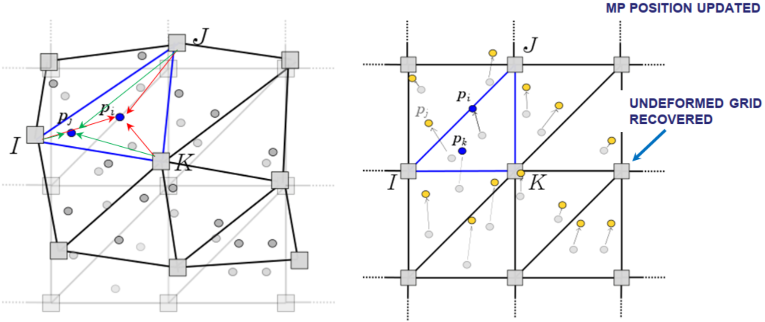
Ω: Initial configuration.
$\varphi(\Omega)$: Current configuration.

The FE traditional steps:

1. Construction of the elemental system. System is evaluated in the current configuration.
2. Assembling of the elemental system.
3. Solving the system.

Stage 3. Convective phase

Information is interpolated and stored on the particles which are moved on the calculated positions.



1. Nodal information is interpolated back onto the material points.
2. MP position is updated.
3. The undeformed FE grid is recovered.
4. The material points connectivities are updated.

Mixed formulation u-p for hyperelastic materials

A standard Galerkin displacement-based formulation fails when Poisson's ratio $\nu \longrightarrow 0.5$ $(\kappa \longrightarrow \infty)$ or when the plastic flow is constrained by the volume conservation condition.

Updated Lagrangian Formulation

$$\begin{cases} \rho \dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - \nabla \cdot \boldsymbol{\sigma} = \rho \boldsymbol{b} & \text{in } \varphi(\Omega)(t), \\ \dfrac{p}{\kappa} - \left(1 - \dfrac{1}{J}\right) = 0 & \text{in } \varphi(\Omega)(t), \\ \boldsymbol{u} = \bar{\boldsymbol{u}} & \text{on } \varphi(\partial\Omega_{\mathrm{D}})(t), \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} & \text{on } \varphi(\partial\Omega_{\mathrm{N}})(t), \end{cases}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{dev}} + p\mathbf{I}$$

Cauchy stress tensor

Hyperelasticity

$$\mathbf{S} = 2\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} \longrightarrow \boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}}.$$

$$\mathbf{S} := 2\frac{\partial \Psi_{\mathrm{dev}}}{\partial \mathbf{C}} + \frac{d\Psi_{\mathrm{vol}}}{dJ} J\mathbf{C}^{-1}$$

- Deviatoric model: Neo-Hookean
- Volumetric model: Miehe et al.

$$\frac{\partial \Psi_{\mathrm{vol}}(J)}{\partial J} = \kappa\left(1 - \frac{1}{J}\right)$$

Mixed formulation for an incompressible Newtonian fluid material

It is a displacement-based formulation, an it is assumed finite strains.

Updated Lagrangian Formulation

$$\begin{cases} \rho \dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - \nabla \cdot \boldsymbol{\sigma} = \rho \boldsymbol{b} & \text{in } \varphi(\Omega)(t), \\ 1 - \dfrac{1}{J} = 0 & \text{in } \varphi(\Omega)(t), \\ \boldsymbol{u} = \bar{\boldsymbol{u}} & \text{on } \varphi(\partial\Omega_{\mathrm{D}})(t), \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} & \text{on } \varphi(\partial\Omega_{\mathrm{N}})(t), \end{cases}$$

$$\boldsymbol{\sigma} = \underbrace{2\mu\nabla^s \boldsymbol{v}}_{\boldsymbol{\sigma}^{\mathrm{dev}}(\boldsymbol{u})} + p\mathbf{I}$$

Cauchy stress tensor

Incompressible Newtonian fluid material

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}}$$

$$\mathbf{S} := J\left(\mu\mathbf{C}^{-1}\dot{\mathbf{C}}\mathbf{C}^{-1}\right) + pJ\mathbf{C}^{-1}$$

where

$$\dot{\mathbf{C}} = \frac{D\mathbf{C}}{Dt} = 2\mathbf{F}^T\nabla^S\boldsymbol{v}\mathbf{F}; \;\; \mathbf{C} = \mathbf{F}^T\mathbf{F}$$

Variational form

The weak form problem consists in finding $\boldsymbol{U} = [\boldsymbol{u}, p] :]0, T[\longrightarrow \mathcal{W}$, such that the initial conditions are satisfied and for all $\boldsymbol{V} = [\boldsymbol{w}, q] \in \mathcal{W}_0$,

$$\left(\mathcal{D}_t\left(\boldsymbol{U}\right), \boldsymbol{V}\right) + B\left(\boldsymbol{U}, \boldsymbol{V}\right) = \mathcal{F}\left(\boldsymbol{V}\right)$$

where:

$$\left(\mathcal{D}_t\left(\boldsymbol{U}\right), \boldsymbol{V}\right) = \left(\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \boldsymbol{w}\right)$$

$$B\left(\boldsymbol{U}, \boldsymbol{V}\right) = \left(\boldsymbol{\sigma}^{\mathrm{dev}}(\boldsymbol{u}), \nabla^{\mathrm{s}} \boldsymbol{w}\right) + \left(p\mathbf{I}, \nabla^{\mathrm{s}} \boldsymbol{w}\right) + \left(\frac{p}{\kappa}, q\right) - \left(1 - \frac{1}{J}, q\right)$$

$$\mathcal{F}\left(\boldsymbol{V}\right) = \langle \rho \boldsymbol{b}, \boldsymbol{w} \rangle + \langle \bar{\boldsymbol{t}}, \boldsymbol{w} \rangle_{\varphi(\partial \Omega_{\mathrm{N}})}$$

Linearization

- Newton-Raphson's iterative procedure. B must allow to compute a correction $\delta \boldsymbol{U} = [\delta \boldsymbol{u}, \delta p]$. $\boldsymbol{U} = \delta \boldsymbol{U} + \boldsymbol{U}^*$.

$$\left(\mathcal{D}_t\left(\delta \boldsymbol{U}\right), \boldsymbol{V}\right) + B_{\mathrm{d}}\left(\delta \boldsymbol{U}, \boldsymbol{V}\right) = \mathcal{F}\left(\boldsymbol{V}\right) - \left(\mathcal{D}_t\left(\boldsymbol{U}^*\right), \boldsymbol{V}\right) - B\left(\boldsymbol{U}^*, \boldsymbol{V}\right)$$

Spatial and temporal discretizations

Spatial discretization of the linearized form.

Galerkin finite element approximation. It consists in finding $\delta \boldsymbol{U}_h \in \boldsymbol{\mathcal{W}}_{h,0}$ such that:

$$\underbrace{\left(\mathcal{D}_t\left(\delta \boldsymbol{U}_h\right), \boldsymbol{V}_h\right)}_{\text{Temporal terms}} + \underbrace{B_{\mathrm{d}}\left(\delta \boldsymbol{U}_h, \boldsymbol{V}_h\right)}_{\text{bilinear form}} = \mathcal{F}\left(\boldsymbol{V}_h\right) - \left(\mathcal{D}_t\left(\boldsymbol{U}_h^*\right), \boldsymbol{V}_h\right) - B\left(\boldsymbol{U}_h^*, \boldsymbol{V}_h\right)$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{W}}_{h,0}$. To discretise the continuum body $\mathcal{B}$ by a set of n_p material points and Ω_p a finite volume of the body to each of those material points.

$$\mathcal{B} \approx \mathcal{B}_h = \bigcup_{p=1}^{n_p} \Omega_p .$$

Time discretization

Monolithic time discretization using a Newmark scheme.

Stabilization technique: Variational Multi-Scale (VMS) Methods

- **Objective**: to approximate the components of the continuous problem solution that cannot be resolved by the finite element mesh.
- Unknown splitting: $\delta \boldsymbol{U} = \underbrace{\delta \boldsymbol{U}_h}_{\in \boldsymbol{\mathcal{W}}_{h,0}} + \underbrace{\tilde{\boldsymbol{U}}}_{\in \tilde{\boldsymbol{\mathcal{W}}}}$ and $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{W}}_{h,0} \bigoplus \tilde{\boldsymbol{\mathcal{W}}}$.

$$\overbrace{\left(\mathcal{D}_t\left(\delta \boldsymbol{U}_h\right), \boldsymbol{V}_h\right) + B_{\mathrm{d}}\left(\delta \boldsymbol{U}_h, \boldsymbol{V}_h\right)}^{\text{Galerkin terms}} + \overbrace{\cancel{\langle \mathcal{D}_t(\tilde{\boldsymbol{U}}), \boldsymbol{V}_h \rangle} + \sum_K \langle \tilde{\boldsymbol{U}}, \overbrace{\mathcal{L}_{\mathrm{d}}^{*}(\boldsymbol{u}_h; \boldsymbol{V}_h)}^{\text{adjoint operator of } \mathcal{L}_{\mathrm{d}}} \rangle_K}^{\text{Stabilization terms}}$$

$$= \underbrace{\mathcal{F}\left(\boldsymbol{V}_h\right) - B\left(\boldsymbol{U}_h^{*}, \boldsymbol{V}_h\right) - \left(\mathcal{D}_t\left(\boldsymbol{U}_h^{*}\right), \boldsymbol{V}_h\right)}_{\text{Galerkin terms}}$$

$$\cancel{\frac{\partial \tilde{\boldsymbol{U}}}{\partial t}} + \boldsymbol{\tau}^{-1} \tilde{\boldsymbol{U}} = \tilde{P} \underbrace{\left[\mathfrak{F} - \mathcal{L}\left(\boldsymbol{U}_h^{*}\right) - \mathcal{D}_t\left(\boldsymbol{U}_h^{*}\right) - \mathcal{D}_t\left(\delta \boldsymbol{U}_h\right) - \mathcal{L}_{\mathrm{d}}\left(\delta \boldsymbol{U}_h\right)\right]}_{\text{Residual from the linearization}}$$

Sub-grid scale

Final expression for linear elements

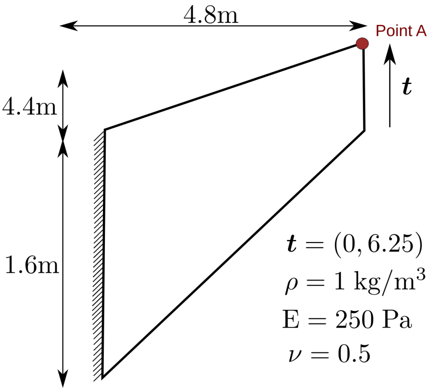
$$\overbrace{\left(\mathcal{D}_t\left(\delta \boldsymbol{U}_h\right), \boldsymbol{V}_h\right) + B_{\mathrm{d}}\left(\delta \boldsymbol{U}_h, \boldsymbol{V}_h\right)}^{\text{Galerkin terms}} + \overbrace{S_1\left(\boldsymbol{U}_h^*; \delta \boldsymbol{U}_h, \boldsymbol{V}_h\right) + S_2\left(\boldsymbol{U}_h^*; \delta \boldsymbol{U}_h, \boldsymbol{V}_h\right)}^{\textbf{Stabilization terms}}$$
$$= \underbrace{\mathcal{F}\left(\boldsymbol{V}_h\right) - B\left(\boldsymbol{U}_h^*, \boldsymbol{V}_h\right) - \left(\mathcal{D}_t\left(\boldsymbol{U}_h^*\right), \boldsymbol{V}_h\right)}_{\text{Galerkin terms}} + \underbrace{R_1\left(\boldsymbol{U}_h^*, \boldsymbol{V}_h\right) + R_2\left(\boldsymbol{U}_h^*, \boldsymbol{V}_h\right)}_{\textbf{Stabilization terms}}$$

$$S_1\left(\boldsymbol{U}_h^*; \delta \boldsymbol{U}_h, \boldsymbol{V}_h\right) =$$
$$\sum_K \tau_1 \left\langle \tilde{P}\left[-\rho \frac{\partial^2 \delta \boldsymbol{u}_h}{\partial t^2} + \left(\nabla \delta \boldsymbol{u}_h \cdot \nabla p_h^*\right) + \left(\nabla \cdot p_h^*\left(\mathbf{I} \otimes \mathbf{I} - 2\mathbb{I}\right) : \nabla^{\mathrm{s}} \delta \boldsymbol{u}_h\right) + \nabla \delta p_h\right], \right.$$
$$\left. -\left(\nabla \boldsymbol{w}_h \cdot \nabla p_h^*\right) - \left(\nabla^{\mathrm{s}} \boldsymbol{w}_h : \nabla \cdot p_h^*\left(\mathbf{I} \otimes \mathbf{I} - 2\mathbb{I}\right)\right) - \mathrm{f}(J(\boldsymbol{u}_h^*)) \nabla q_h \right\rangle_K$$

$$S_2\left(\boldsymbol{U}_h^*; \delta \boldsymbol{U}_h, \boldsymbol{V}_h\right) = \sum_K \tau_2 \left\langle \tilde{P}\left[-\frac{\delta p_h}{\kappa} + \mathrm{f}(J(\boldsymbol{u}_h^*)) \nabla \cdot \delta \boldsymbol{u}_h\right], \nabla \cdot \boldsymbol{w}_h + \frac{q_h}{\kappa} \right\rangle_K$$

- $\tilde{P}$ is the L^2 – projection onto the space of sub-grid scales,
- $\boldsymbol{\tau}$ is a matrix computed within each element

Cook's membrane. Main features.

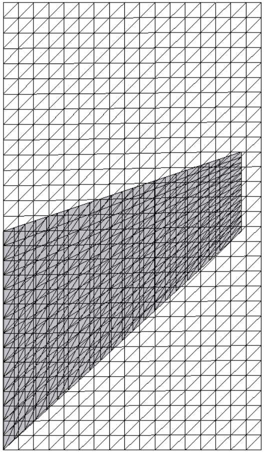


$\boldsymbol{t} = (0, 6.25)$ Pa

$\rho = 1$ kg/m^3

E = 250 Pa

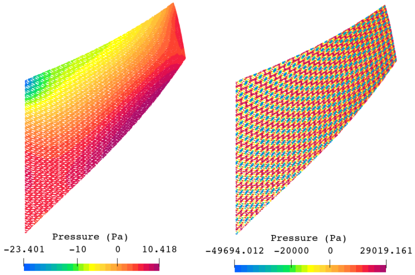
$\nu = 0.5$



Temporal discretization: $\delta t = 0.0025$ s

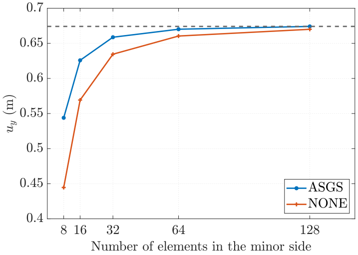
Spatial discretization: $h = 0.6, 0.3, 0.15, 0.75, 0.325$ m with 16 material points per element.

Cook's membrane: static and dynamic cases.

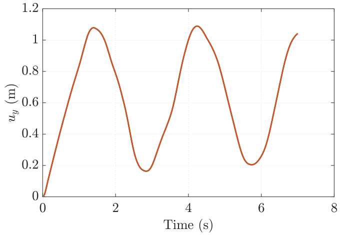


Pressure using stabilization VS without stabilization.

- Pressure distribution without jumps between elements.
- Dynamic case is impossible to run without stabilization.

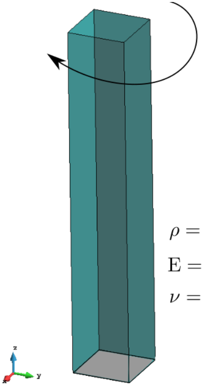


Mesh convergence. Static case.

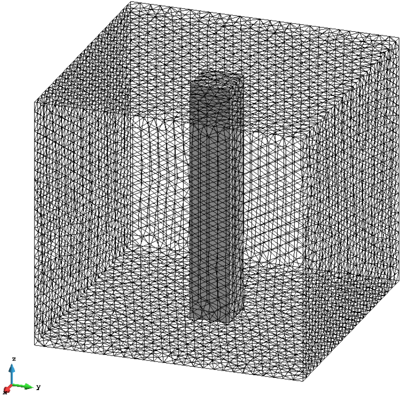


Evolution of the displacement in time. Dynamic case.

Twisting column problem. Main features



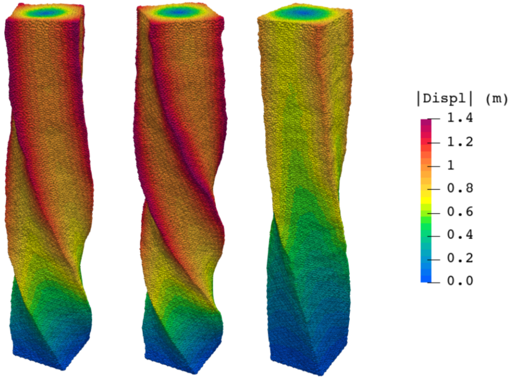
$\rho = 1100\ \mathrm{kg/m^3}$
$\mathrm{E} = 1.7 \times 10^7\ \mathrm{Pa}$
$\nu = 0.5$



$$\boldsymbol{v}^0(x, y, z) = 105 \sin\left(\frac{\pi z}{12}\right)(y, -x, 0)^T \ \mathrm{m/s}$$

Temporal discretization: $\delta t = 0.001$ s, $T_{\mathrm{fin}} = 0.5$ s,
Spatial discretization: mesh size $h = 0.2$ m, and 16 material points per element in the body.

Twisting column problem



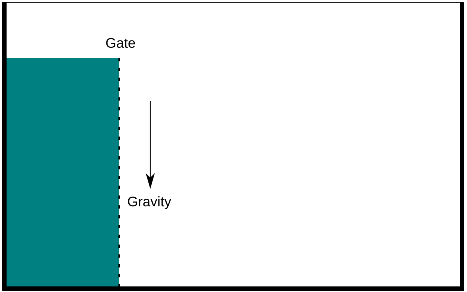
Twisting column at $t = 0.05$ s, $t = 0.1$ s and $t = 0.2$ s.

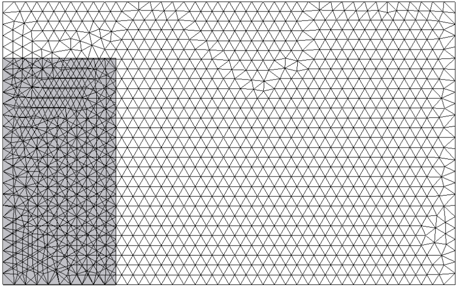
- Simulation is done without remeshing techniques, ALE or level set.

Conclusion

ASGS stabilization method is **more robust than other methods**, such as Polynomial Pressure Method (PPP).

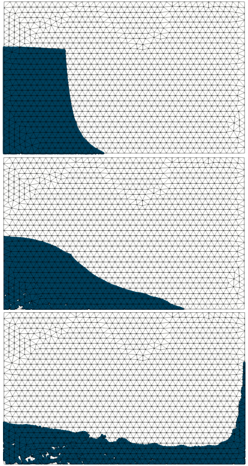
Dam break problem. Main features



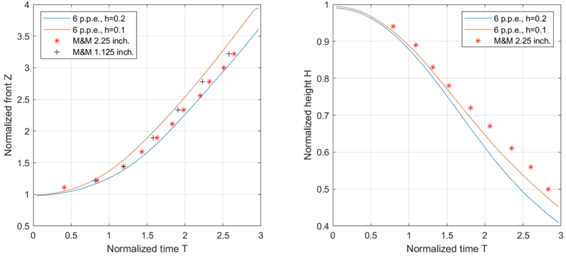


Spatial discretization: Mesh size $h = 0.1$ m and $h = 0.2$ m, and 6 material points per element.
Temporal discretization: $\delta t = 0.001\ \mathrm{s}$, $T_{\mathrm{fin}} = 0.9\ \mathrm{s}$.

Dam break problem. Results



Dam break test at t=0.25, 0.5 and 0.85 s.



Left: distance of the surge front from axis of plane of symmetry; Right: height of the residual column.

- **Volumetric locking** is avoided.
- In agreement with **experimental evidence**.

Conclusions

- VMS-type stabilization have been developed and implemented for a **solid mechanics framework** and the Material Point Method.
- **ASGS stabilization** is more robust in comparison with others stabilization techniques such as the Polynomial Pressure Projection.
- It is able to compute very challenging **large deformation regimes**.
- **Newtonian law** for an Updated Lagrangian linearized displacement-based formulation have been developed. It is checked using an experimental test.

References

1. Iaconeta, I. et al. A stabilized mixed implicit material point method for non-linear incompressible solid mechanics. *Computational Mechanics* 63.6 (2019): 1243-1260.
2. Contri, A. An Updated Lagrangian displacement-based formulation for free surface incompressible fluids using MPM. Master Thesis.
3. Castañar I., Baiges J. and Codina R. A stabilized mixed finite element approximation for incompressible finite strain solid dynamics using a total Lagrangian formulation. *Computer Methods in Applied Mechanics and Engineering* 368 (2020): 113164.
4. Hassan, Osama I. et al. An upwind vertex centred finite volume algorithm for nearly and truly incompressible explicit fast solid dynamic applications: Total and Updated Lagrangian formulations. *Journal of Computational Physics*: X 3 (2019): 100025.
5. Cervera, M. et al. Mixed stabilized finite element methods in nonlinear solid mechanics. Part III: Compressible and incompressible plasticity. *Computer Methods in Applied Mechanics and Engineering* 285 (2015): 752-775.
6. Martin J. C., Moyce W. J., Penney W. G., Price A. T. Thornhill C. K. (1952). Part IV. An experimental study of the collapse of liquid columns on a rigid horizontal plane. Philosophical Transactions of the Royal Society of London. Series A, *Mathematical and Physical Sciences*, 244(882), 312-324.

Università degli Studi di Padova

September 2022

Thank you for your attention!!

Laura Moreno Martínez

Linearization

- Newton-Raphson's iterative procedure.
- Taylor's series expansion evaluated at the last known equilibrium configuration $\boldsymbol{U}^* = [\boldsymbol{u}^*, p^*]$.
- B must be allows to compute a correction $\delta\boldsymbol{U} = [\delta\boldsymbol{u}, \delta p]$. Note that $\boldsymbol{U} = \delta\boldsymbol{U} + \boldsymbol{U}^*$.

$$B(\boldsymbol{U}, \boldsymbol{V}) \approx B(\boldsymbol{U}^*, \boldsymbol{V}) + B_{\mathrm{d}}(\delta\boldsymbol{U}, \boldsymbol{V}) + o(\delta\boldsymbol{u}) + o(\delta p)$$

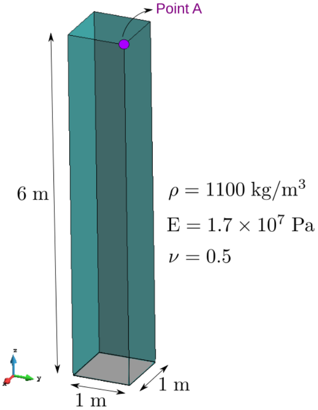
Find the correction $\delta\boldsymbol{U} = [\delta\boldsymbol{u}, \delta p] :]0, T[\longrightarrow \boldsymbol{\mathcal{W}}_0$ such that

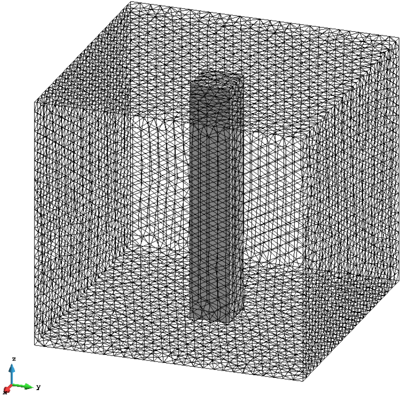
$$\boxed{(\mathcal{D}_t(\delta\boldsymbol{U}), \boldsymbol{V}) + B_{\mathrm{d}}(\delta\boldsymbol{U}, \boldsymbol{V}) = \mathcal{F}(\boldsymbol{V}) - (\mathcal{D}_t(\boldsymbol{U}^*), \boldsymbol{V}) - B(\boldsymbol{U}^*, \boldsymbol{V})}$$

$$\mathcal{D}_t(\delta\boldsymbol{U}) = \left[\rho\frac{\partial^2\delta\boldsymbol{u}}{\partial t^2}, 0\right]$$

$$\begin{aligned} B_{\mathrm{d}}(\delta\boldsymbol{U}, \boldsymbol{V}) = & (\nabla\delta\boldsymbol{u}\cdot(\boldsymbol{\sigma}(\boldsymbol{u}^*) + p^*\mathbf{I}), \nabla\boldsymbol{w}) \\ & + \left(\nabla^{\mathrm{s}}\boldsymbol{w}, \mathbb{C}^{\mathrm{dev}}(\boldsymbol{u}^*) + p^*(\mathbf{I}\otimes\mathbf{I} - 2\mathbb{I}) : \nabla^{\mathrm{s}}\delta\boldsymbol{u}\right) \\ & + (\delta p, \nabla\cdot\boldsymbol{w}) + \left(\frac{\delta p}{\kappa}, q\right) - (\mathrm{f}(J(\boldsymbol{u}^*))\nabla\cdot\delta\boldsymbol{u}, q) \end{aligned}$$

Bending beam problem. Main features

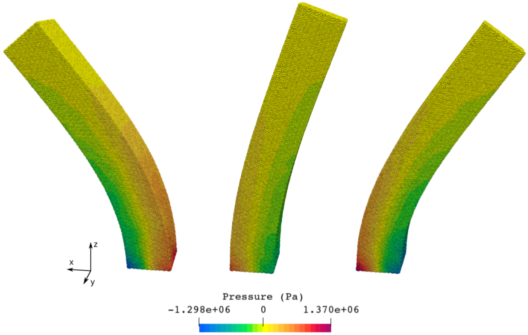




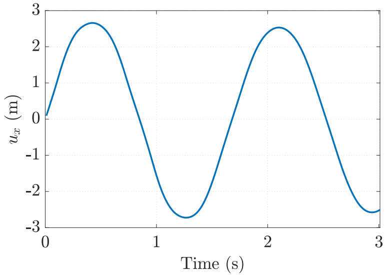
$$\boldsymbol{v}^0(x, y, z) = \frac{5}{3}(z, 0, 0)^T\ \mathrm{m/s}$$

Temporal discretization: $\delta t = 0.01$ s, $T_{\text{fin}} = 3$ s,
Spatial discretization: mesh size $h = 0.2$ m, and 16 material points per element in the body.

Bending beam problem. Results



Bending column at $t = 0.5$ s, $t = 1$ s and $t = 1.25$ s.



Evolution of x-displacement in time of the point A.