UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

CIMNE

25th International Congress of Theoretical and Applied Mechanics.

# Solution of transient viscoelastic flow problems approximated by a VMS stabilized finite element formulation using time-dependent subrid-scales

Laura Moreno Martínez
Advisors: Ramon Codina and Joan Baiges

August, 2021

## Dynamic Subscales: Why are these necessary?

Classical residual-based stabilized methods for unsteady incompressible flows may experience **difficulties when the time step is small** relative to the spatial grid size.

- Bochev et al. argue that $\delta t > Ch^2$ **a sufficient condition** to avoid instabilities.
- For anisotropic space-time discretizations, this inequality **is not necessarily satisfied**: a very common issue in viscoelastic flow formulations.

**Consequently...**

**New stabilization techniques must be designed** to compute time-dependent viscoelastic flow problems with high elasticity and anisotropic space-time discretization.

# Logarithmic conformation formulation

## Main features

- High Weissenberg Number Problem: All numerical methods found a **breakdown** at a low Weissenberg number.
- Log-conformation formulation was proposed by **Fattal and Kupferman** (2004).
- Treats the **exponential growth** of the elastic stresses when the elastic component becomes dominant.
- Allows to **extend the range** of Weissenberg numbers. ✓
- More computational **expensive** than the standard formulation. ✗

- Physically-admissible conformation tensors must be **symmetric** and **positive-definite**.

Elastic stress tensor

$$\boldsymbol{\sigma} = \frac{\eta_p}{\lambda}(\boldsymbol{\tau} - \mathbf{I}) \longrightarrow \boldsymbol{\tau} = \frac{\lambda \boldsymbol{\sigma}}{\eta_p} + \mathbf{I}$$

Conformation tensor

- Conformation tensor is replaced by $\boxed{\psi = \log(\boldsymbol{\tau}).}$

# Spatial and temporal discretization

- **Galerkin FE** approximation. Consists in finding $\boldsymbol{U}_h : (0, t_\mathrm{f}) \longrightarrow \boldsymbol{\mathcal{X}}_h$,

$$\underbrace{(\mathcal{D}_t(\boldsymbol{U}_h), \boldsymbol{V}_h)}_{\text{Temporal terms}} + \underbrace{B(\boldsymbol{u}; \boldsymbol{U}_h, \boldsymbol{V}_h)}_{\text{Bilinear form}} = L(\boldsymbol{V}_h),$$

  for all $\boldsymbol{V}_h = [\boldsymbol{v}_h, q_h, \chi_h] \in \boldsymbol{\mathcal{X}}_h$

- Monolithic **time discretization**. BDF1 and BDF2 schemes have been employed in the work.

# Stabilized formulation: Variational multiscale methods

- To approximate the components of the continuous problem solution that cannot be resolved by the finite element mesh.
- Split the unknowns as $\boldsymbol{U} = \underbrace{\boldsymbol{U}_h}_{\in \boldsymbol{\mathcal{X}}_h} + \underbrace{\tilde{\boldsymbol{U}}}_{\in \tilde{\boldsymbol{\mathcal{X}}}}$ and $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}_h \bigoplus \tilde{\boldsymbol{\mathcal{X}}}$.

$$\tilde{\boldsymbol{U}} = \boldsymbol{\alpha} \tilde{P}[\mathbf{F} - \mathcal{D}_t(\boldsymbol{U}_h) - \mathcal{L}(\boldsymbol{u}_h; \boldsymbol{U}_h)]$$

Sub-grid scale

$$\underbrace{\underbrace{(\mathcal{D}_t(\boldsymbol{U}_h), \boldsymbol{V}_h)}_{\text{Temporal terms}} + \underbrace{B(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h)}_{\text{Bilinear terms}}}_{\text{Galerkin terms}} + \underbrace{\sum_K \langle \tilde{\boldsymbol{U}}, \underbrace{\mathcal{L}^*(\boldsymbol{u}_h; \boldsymbol{V}_h)}_{\text{adjoint operator of } \mathcal{L}} \rangle_K}_{\text{Stabilization terms}} = L(\boldsymbol{V}_h)$$

- $\tilde{P}$ is the $L^2$ projection onto the space of sub-grid scales,
- $\boldsymbol{\alpha}$ is a matrix computed within each element,
- $\mathcal{L}$ is the operator associated to the problem.

# Stabilized formulation: dynamic subscales. Adding a temporal derivative

$$\underbrace{\underbrace{(\mathcal{D}_t(\boldsymbol{U}_h), \boldsymbol{V}_h)}_{\text{Temporal terms}} + \underbrace{B(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h)}_{\text{Bilinear terms}}}_{\text{Galerkin terms}} + \underbrace{\sum_K \langle \tilde{\boldsymbol{U}}, \underbrace{\mathcal{L}^*(\boldsymbol{u}_h; \boldsymbol{V}_h)}_{\text{adjoint operator of } \mathcal{L}} \rangle_K}_{\text{Stabilization terms}} = L(\boldsymbol{V}_h)$$

- Quasi-static subscales:

$$\boldsymbol{\alpha}^{-1}\tilde{\boldsymbol{U}} = \tilde{P}[\boldsymbol{F} - \mathcal{D}_t(\boldsymbol{U}_h) - \mathcal{L}(\boldsymbol{u}_h; \boldsymbol{U}_h)]$$

- Dynamic subscales:

$$\boxed{\frac{\partial \tilde{\boldsymbol{U}}}{\partial t} + \boldsymbol{\alpha}^{-1}\tilde{\boldsymbol{U}} = \tilde{P}[\boldsymbol{F} - \mathcal{D}_t(\boldsymbol{U}_h) - \mathcal{L}(\boldsymbol{u}_h; \boldsymbol{U}_h)]}$$

# Stabilized formulation: Residual based vs Split OSS.

$$\underbrace{\mathcal{D}_t(\boldsymbol{U}_h),\boldsymbol{V}_h) + B(\boldsymbol{u}_h;\boldsymbol{U}_h,\boldsymbol{V}_h)}_{\text{Galerkin terms}} + \overbrace{S_1(\boldsymbol{u}_h;\boldsymbol{U}_h,\boldsymbol{V}_h) + S_2(\boldsymbol{U}_h,\boldsymbol{V}_h) + S_3(\boldsymbol{u}_h;\boldsymbol{U}_h,\boldsymbol{V}_h)}^{\text{Stabilization terms}} = \underbrace{L(\boldsymbol{V}_h)}_{\text{Gal. term}} + \overbrace{R_1(\boldsymbol{u}_h) + R_3(\boldsymbol{u}_h)}^{\text{Stab. term}}$$

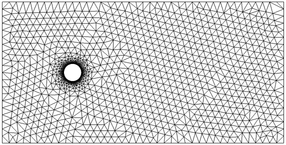
$$S_1(\hat{\boldsymbol{u}}_h;\boldsymbol{U}_h,\boldsymbol{V}_h) = \sum_K \alpha_{1\text{dyn}} \Big\langle \rho \frac{1}{\delta t}\tilde{\boldsymbol{u}}^n - \tilde{P}\Big[\rho\frac{\partial \boldsymbol{u}_h}{\partial t} - \frac{\eta_p}{\lambda_0}\nabla\cdot(\exp(\boldsymbol{\psi}_h)) - 2\eta_e\nabla\cdot(\nabla^s\boldsymbol{u}_h) + \rho\hat{\boldsymbol{u}}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h\Big],$$
$$-\nabla\cdot\boldsymbol{\chi}_h + 2\eta_e\nabla\cdot(\nabla^s\boldsymbol{v}_h) + \rho\hat{\boldsymbol{u}}_h\cdot\nabla\boldsymbol{v}_h + \nabla q_h\Big\rangle_K$$

$$\tilde{P} = P_h^{\perp}$$

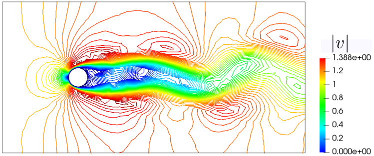
Split OSS

1. The **residual** based stabilization contemplates all terms.
2. **Split** OSS stabilization: neglect the cross local inner-product terms as well as some other terms that do not contribute to stability.

# Example: Flow over a cylinder. Comparing stabilizations



Coarse mesh



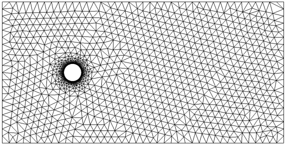

$\delta t = 0.1$

| **P1 elements** | **Time step ($\delta t$)** | | | |
|---|---|---|---|---|
| **Method** | 0.050 | 0.0250 | $3.125 \times 10^{-3}$ | $1.562 \times 10^{-3}$ |
| Static-OSS | Solved | Failed | - | - |
| Dyn-OSS | Solved | Solved | Solved | Solved |
| Static-SOSS | Solved | Solved | Solved | Failed |
| Dyn-SOSS | Solved | Solved | Solved | Solved |

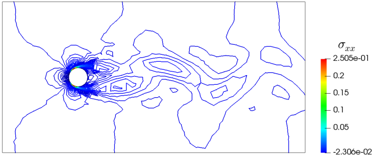
Table: Solved and failed cases $\mathrm{We} = 0.125$, $\alpha_{1,\min} \approx 1.156 \times 10^{-3}$.

The most unstable stabilization is the **quasi-static** + OSS stabilization.

# Example: Flow over a cylinder. Comparing formulations



Coarse mesh



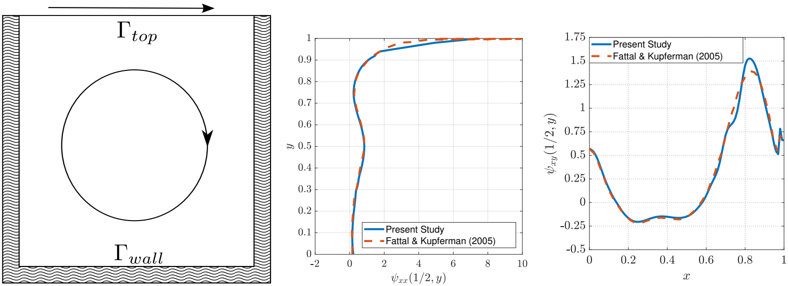

$\delta t = 0.1$

| | Weissenberg (We) | | | |
|---|---|---|---|---|
| **Formulation** | 0.125 | 0.165 | 0.25 | 0.5 |
| Std-Static | Solved | Failed | - | - |
| Std-Dyn | Solved | Solved | Solved | Failed |
| Log-Static | Solved | Solved | Failed | - |
| Log-Dyn | Solved | Solved | Solved | Solved |

Table: Solved and failed cases for S-OSS formulations, dynamic and quasi-static, $\delta t = 0.1$.

**Dynamic formulations** are more efficient avoiding elastic instabilities.

# Benchmark: Lid-driven cavity flow problem. Case Re=0.




| | Stabilization S-OSS | |
|---|---|---|
| **Formulation** | Quasi-static | Dynamic |
| Standard | Failed - time step 265 | Failed - time step 1316 |
| Logarithmic | Failed - time step 340 | Solved |

Table: Comparison between different formulations, $\mathrm{We} = 1.0$, $\delta t = 0.0025$. The time step at which convergence fails is indicated.

# References

**Presentation based on the paper:**


- L. Moreno, R. Codina and J.Baiges. Solution of transient viscoelastic flow problems aproximated by a term-by-term VMS stabilized finite element formulation using time-dependent subgrid-scales. Computer Methods in Applied Mechanics and Engineering 367 (2020): 113074.

# References


- R. Fattal and R. Kupferman. Constitutive laws for the matrix-logarithm of the conformation tensor. Journal of Non-Newtonian Fluid Mechanics, 123(2-3):281–285, 2004.
- E. Castillo and R. Codina. Variational multi-scale stabilized formulations for the stationary three-field incompressible viscoelastic flow problem. Computer Methods in Applied Mechanics and Engineering, 279:579–605, 2014.
- M. A Hulsen, R. Fattal, and R. Kupferman. Flow of viscoelastic fluids past a cylinder at high weissenberg number: stabilized simulations using matrix logarithms. Journal of Non-Newtonian Fluid Mechanics, 127(1):27–39, 2005.
- P. B. Bochev, M. D. Gunzburger, and J. N. Shadid. "On inf–sup stabilized finite element methods for transient problems." Computer Methods in Applied Mechanics and Engineering 193.15-16: 1471-1489, 2004.
- E. Castillo and R. Codina. Dynamic term-by-term stabilized finite element formulation using orthogonal subgrid-scales for the incompressible Navier–Stokes problem. Computer Methods in Applied Mechanics and Engineering, 349, 701-721. 2019

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

CIMNE

25th International Congress of Theoretical and Applied Mechanics

August 2021

## Thank you for your attention!!

Laura Moreno Martínez