Solution of transient viscoelastic flow problems approximated by a VMS stabilized finite element formulation using time-dependent subrid-scales

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Abstract

Recent studies indicate that classical residual-base stabilized methods for unsteady incompressible flows may experience difficulties when the time step is small in relation with the spatial grid size. The aim of this work is the design of finite element stabilized techniques based on the Variational Multiscale (VMS) method that allow to compute time-dependent viscoelastic flow problems with high elasticity and considering an anisotropic space-time discretization. Although the main advantage is achieve stable solutions for these discretizations, other benefits related with elastic problems are proved in this study.

Introduction

Classical residual-based stabilized methods for unsteady incompressible flows may experience **difficulties when the time step is small** relative to the spatial grid size.

- Bochev et al. [1] argue that $\delta t > Ch^2$ is **a sufficient condition** to avoid instabilities.
- For anisotropic space-time discretizations, this inequality **is not necessarily satisfied**: a very common issue in viscoelastic flow formulations argued by [2].

New stabilization techniques must be designed to compute time-dependent viscoelastic flow problems with high elasticity and anisotropic space-time discretization. The present work pursues to expand transient subgrid-scale methods to the viscoelastic flow problem, such as it is presented in [3] for the Navier-Stokes incompressible problem using a split term-by-term method.

The computation of viscoelastic flows leads to its own difficulties, **when elasticity becomes dominant**. Therefore, authors employ also the **log-conformation reformulation** using a stabilized formulation based on the VMS method to deal with these shortcomings [4].

Viscoelastic fluid flow problem

The governing equations for the viscoelastic flow problem in incompressible and isothermal conditions, are the conservation of momentum and mass and a constitutive equation:

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla\cdot\boldsymbol{T} + \nabla p = \boldsymbol{f}, \qquad (1)$$

$$\nabla\cdot\boldsymbol{u} = 0, \qquad (2)$$

$$\frac{1}{2\eta_p}\boldsymbol{\sigma} - \nabla^s\boldsymbol{u} + \frac{\lambda}{2\eta_p}\left(\frac{\partial\boldsymbol{\sigma}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\nabla\boldsymbol{u} - (\nabla\boldsymbol{u}^T)\cdot\boldsymbol{\sigma}\right) = \boldsymbol{0}, \qquad (3)$$

The domain considered Ω of $\mathbb{R}^d$ (d=2 or 3) during the time interval $[0, t_\mathrm{f}]$. ρ is the constant density, p is the pressure field, $\boldsymbol{u}$ is the velocity field, $\boldsymbol{f}$ is the force field and $T = 2\eta_s\nabla^s\boldsymbol{u}+\boldsymbol{\sigma}$ is the deviatoric stress tensor. Solvent viscosity is η_s and the polymeric viscosity by η_p, and λ is the relaxation time.

The logarithmic reformulation of the equations is derived basically from a change of variables, see complete development employed is extensively explained in [4].

Stabilization based on VMS

The stabilization method departs from the framework described in [5]. Let us suppose that $\mathcal{L}(\hat{\boldsymbol{u}};\cdot)$ is the linear operator associated to the problem. Introducing the sub-grid scale decomposition and integrating by parts, the method leads to find $\boldsymbol{U}_h :]0, t_\mathrm{f}[\longrightarrow \boldsymbol{\mathcal{X}}_h$ such that

$$\mathcal{G}(\boldsymbol{U}_h, \boldsymbol{V}_h) + B(\boldsymbol{u}_h; \boldsymbol{U}_h, \boldsymbol{V}_h) + \sum_K \langle\tilde{\boldsymbol{U}}, \mathcal{L}^*(\boldsymbol{u}_h; \boldsymbol{V}_h)\rangle_K = L(\boldsymbol{V}_h), \qquad (4)$$

for all $\boldsymbol{V}_h \in \boldsymbol{\mathcal{X}}_h$, where B is the bilinear form of the problem and $\mathcal{G}$ the temporal terms, $\mathcal{L}^*(\boldsymbol{u}_h; \boldsymbol{V}_h)$ is the formal adjoint of the operator $\mathcal{L}(\hat{\boldsymbol{u}};\,\cdot\,)$ and $\tilde{\boldsymbol{U}}$ is the sub-grid scale.

Once operators $\mathcal{G}$ and $\mathcal{L}$ are defined for both formulations, the **sub-grid scales are now the solution of this equation**:

$$\mathcal{D}_t(\tilde{\boldsymbol{U}}) + \boldsymbol{\alpha}^{-1}\tilde{\boldsymbol{U}} = \tilde{P}[\boldsymbol{F} - \mathcal{G}(\boldsymbol{U}_h) - \mathcal{L}(\boldsymbol{u}_h; \boldsymbol{U}_h)], \qquad (5)$$

where $\tilde{P}$ is the L^2 projection onto the space of sub-grid scales, $\mathcal{D}_t(\tilde{\boldsymbol{U}})$ is defined as the temporal derivative of the sub-grid scale and $\boldsymbol{\alpha}$ is taken as a diagonal matrix of stabilization parameters.

Apart from the purely **residual-based stabilization** (denoted by OSS), we propose a **term-by term stabilization** motivated by the fact that not all the terms of some products provide stability, denoted by S-OSS.

The idea of this method is to replace the solution of (5) in (4) but keep only the terms of the form one operator term applied to the unknown by the same operator term applied to the test function, thus neglecting the products of different operators.

Numerical Results

Flow over a cylinder

The flow over a cylinder problem is used to achieve several objectives: firstly, to **compare the various stabilization methods** proposed (dynamic and quasi-static formulations) in terms of stability when the time step is small, and when the Weissenberg number increases.

P1 elements	**Time step (δt)**			
Method	0.050	0.0250	3.125×10^{-3}	1.562×10^{-3}
Static-OSS	Solved	Failed	-	-
Dyn-OSS	Solved	Solved	Solved	Solved
Static-SOSS	Solved	Solved	Solved	Failed
Dyn-SOSS	Solved	Solved	Solved	Solved

Table: Solved and failed cases We $= 0.125$,$\alpha_{1,\mathrm{min}} \approx 1.156\times10^{-3}$. The dynamic method is the most efficient

	Weissenberg (We)			
Formulation	0.125	0.165	0.25	0.5
Std-Static	Solved	Failed	-	-
Std-Dyn	Solved	Solved	Solved	Failed
Log-Static	Solved	Solved	Failed	-
Log-Dyn	Solved	Solved	Solved	Solved

Table: Solved and failed cases for S-OSS formulations,dynamic and quasi-static, $\delta t = 0.1$. Dynamic formulations are more effective avoiding elastic instabilities.

Lid driven cavity

The lid-driven cavity flow is a good example to illustrate the differences that can be generated by the viscoelastic contribution in the fluid, due to the elastic stresses dependence on the previous deformation history. In this case, we have solved it to prove that the **dynamic term-by-term formulation is also efficient**.

	Stabilization S-OSS	
Formulation	Quasi-static	Dynamic
Standard	Failed - time step 265	Failed - time step 1316
Logarithmic	Failed - time step 340	Solved

Table: Comparison between different formulations, Re $= 0$,We $= 1.0$, $\delta t = 0.0025$. The time step at which convergence fails is indicated.

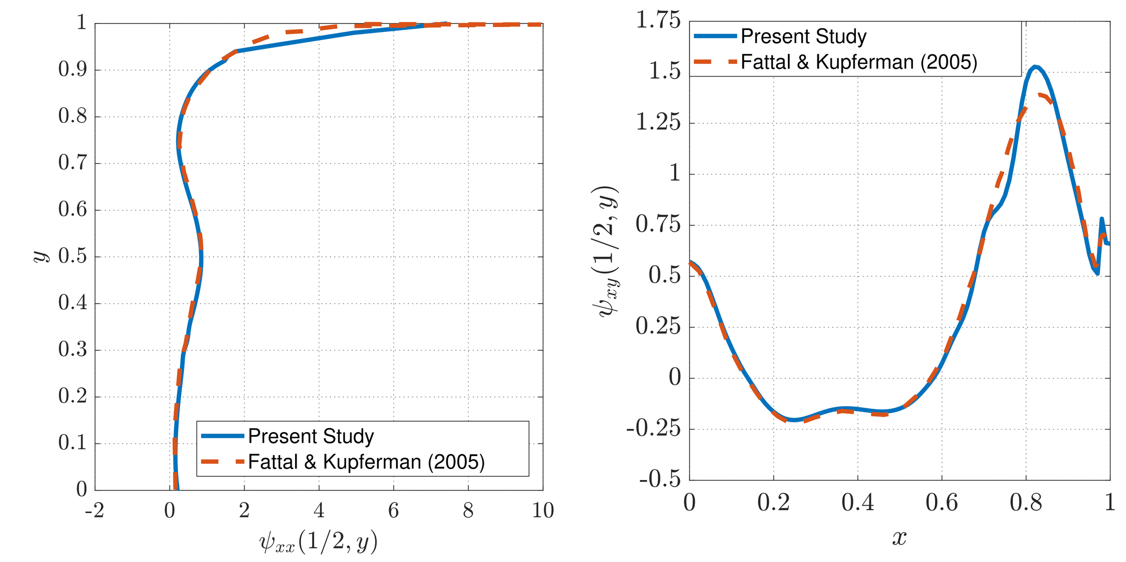

Figure: Results at time $t = 8$, for We $= 1$. $\boldsymbol{\psi}$ profiles along the lines $x = 1/2$ and $y = 3/4$.

Conclusions

- **Dynamic sub-scales** allow to solve problems where two different sources of instability can appear simultaneously: one originated by a time step small and the other the exponential growth typical of high Weissenberg numbers.
- The resulting method allows to obtain **globally stable solutions**, validated in different benchmarks.
- It shows **accuracy**, optimal convergence for smooth solutions and robustness.
- Results are remarkable due to the **Weissenberg number reached** with the dynamic formulation, apart from evident benefits in anisotropic space-time discretizations when the time step is small.
- Combination logarithmic formulation and dynamic subscales in term-by-term stabilization is capable of solving problems with **higher elasticity** than other options.

References

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